Non-spurious solutions to discrete boundary value problems through variational methods

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Abstract

In this talk using direct variational method we investigate the existence of non-spurious solutions to the following Dirichlet problem

$$\ddot{x}\left(t\right)=f\left(t,x\left(t\right)\right),\ x\left(0\right)=x\left(1\right)=0 \tag{1}$$

considered in $H_0^1\left(0,1\right)$, where $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a jointly continuous function convex in x and which does not need to satisfy any further growth conditions. There have been some research in this case addressing mainly problems whose solutions where obtained by the fixed point theorems and the method of lower and upper solutions. We present a first attempt by variational approach.

We now make precise what is meant by a non-spurious solution. For a fixed $n\in\mathbb{N}$ we investigate the numerical approximation to by (1) with $k\in\mathbb{N}(0,n-1)$

$$\Delta^2x\left(k-1\right)=\frac{1}{n^2}f\left(\frac{k}{n},x\left(k\right)\right),\ x\left(0\right)=x\left(n\right)=0. \tag{2}$$

Here Δ is the forward difference operator. Assume that both (1) and (2) for each fixed $n\in\mathbb{N}$ are uniquely solvable by, respectively x and $x^n=(x^n(k))$. Moreover, let there exist two constants $Q,N>0$ independent of n and such that

$$n|\Delta x^n(k-1)|\leq Q \text{ and } |x^n(k)|\leq N$$

for all $k\in\mathbb{N}(0,n)$ and all $n\geq n_0$, where n_0 is fixed (and arbitrarily large). Then it holds

$$\lim_{n\to\infty}\max_{0\leq k\leq n}\left|x^n\left(k\right)-x\left(\frac{k}{n}\right)\right|=0. \tag{3}$$

In other words, this means that the suitable chosen discretization approaches the given continuos boundary value problem. The solutions of a family of problems (2) which converge to some solution of (1) in the sense described by relation (3) are addressed as non-spurious solutions.

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