ω-Attractors and their Basins

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Dynamical systems on a compact space X, generated by a continuous map $f : X \to X$, are considered. For every $x \in X$, the asymptotic behavior of the trajectory $f^i(x), i = 0, 1, 2, ...$, is usually characterized by its ω-limit set $\mathcal{A}_x = \bigcap_{m>0} \overline{\{\bigcup_{i>m} f^i(x)\}}$. Each of the sets $\mathcal{A}_x$, $x \in X$, can be an ω-limit set for many other trajectories. It therefore makes sense to call any set which is an ω-limit set at least for one trajectory an *ω-attractor*. As is well known, if a dynamical system has the so-called Smale horseshoe, then it has very many ω-attractors.

The talk deals with the properties of the dynamical system on an ω-attractor and the properties of the set of all ω-attractors as a set in the space 2^X of all nonempty closed subsets of X, endowed with the Hausdorff metric. In particular, it is known that if X is an interval I, then the set of all ω-attractors is always closed but very complicated, where f has a cycle of period $\neq 2^k, k \geq 0$. In this case for a certain $m > 0$, the map f^m has a one-dimensional horseshoe (or Λ-scheme), namely, there are two disjoint intervals J_1, J_2 such that $f^m(J_1), f^m(J_2) \supset J_1 \cup J_2$, and then the dynamical system, for example, has continuum many different minimal ω-attractors, each being a Cantor set.

The set of trajectories attracted by an ω-attractor is called the *basin of the ω-attractor*, i.e. if $\mathcal{A}$ is an ω-attractor, then $\mathfrak{B}(\mathcal{A}) = \{x \in X \mid \mathcal{A}_x = \mathcal{A}\}$ is a basin of $\mathcal{A}$. As is known $\mathfrak{B}(\mathcal{A})$ is always an $F_{\sigma\delta}$-set, i.e. it can be represented in the form $\bigcap_k \bigcup_j F_{jk}$ with F_{jk} being closed sets, but this upper estimate of the basin complexity is achieved even in the case $X = I$: if an ω-attractor $\mathcal{A}$ contains a cycle and any neighborhood of $\mathcal{A}$ contains an ω-attractor $\widetilde{\mathcal{A}} \supset \mathcal{A}$, then $\mathfrak{B}(\mathcal{A})$ is not a $G_{\delta\sigma}$-set, and hence $\mathfrak{B}(\mathcal{A})$ is a set of the third Baire class.

Most of the results presented in the talk were obtained back in the 60s of the last century, but they are not well known. In the talk we discuss the structure of the basins of ω-attractors as well as some unsolved problems.